

# Integrable structure of melting crystal model with external potentials

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## Abstract

This is a review of the authors' recent results on an integrable structure of the melting crystal model with external potentials. The partition function of this model is a sum over all plane partitions (3D Young diagrams). By the method of transfer matrices, this sum turns into a sum over ordinary partitions (Young diagrams), which may be thought of as a model of  $q$ -deformed random partitions. This model can be further translated to the language of a complex fermion system. A fermionic realization of the quantum torus Lie algebra is shown to underlie therein. With the aid of hidden symmetry of this Lie algebra, the partition function of the melting crystal model turns out to coincide, up to a simple factor, with a tau function of the 1D Toda hierarchy. Some related issues on 4D and 5D supersymmetric Yang-Mills theories, topological strings and the 2D Toda hierarchy are briefly discussed.

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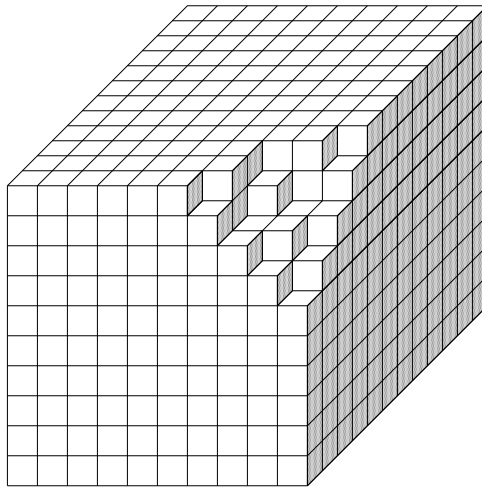


Figure 1: Melting crystal corner

## 1 Introduction

The melting crystal model is a model of statistical mechanics that describes a melting corner of a semi-infinite crystal (Figure1). The crystal is made of unit cubes, which are initially placed at regular positions and fills the positive octant  $x, y, z \geq 0$  of the three dimensional Euclidean space. As the crystal melts, a finite number of cubes are removed from the corner. The present model excludes such crystals that have “overhangs” viewed from the  $(1, 1, 1)$  direction. In other words, the complement of the crystal in the positive octant is assumed to be a 3D analogue of Young diagrams (Figure2). Since 3D Young diagrams are represented by “plane partitions”, the melting crystal model is also referred to as a model of “random plane partitions”.

Though combinatorics of plane partitions has a rather long history [1], Okounkov and Reshetikhin [2] proposed an entirely new approach in the course of their study on a kind of stochastic process of random partitions (the Schur process). Their approach was based on “diagonal slices” of 3D Young diagrams and “transfer matrices” between those slices. As a byproduct, they could re-derive a classical result of MacMahon [1] on the generating function of the numbers of plane partitions. Actually, this generating function is nothing but the partition function of the aforementioned melting crystal model. The method of Okounkov and Reshetikhin was soon generalized [3] to deal with the topological vertex [4, 5] of  $A$ -model topological strings on toric Calabi-Yau threefolds.

The melting crystal model is also closely related to supersymmetric gauge theories. Namely, with slightest modification, the partition function can be interpreted as the instanton sum of 5D  $\mathcal{N} = 1$  supersymmetric (SUSY)  $U(1)$  Yang-Mills theory on partially compactified space-time  $\mathbf{R}^4 \times S^1$  [6]. This in-

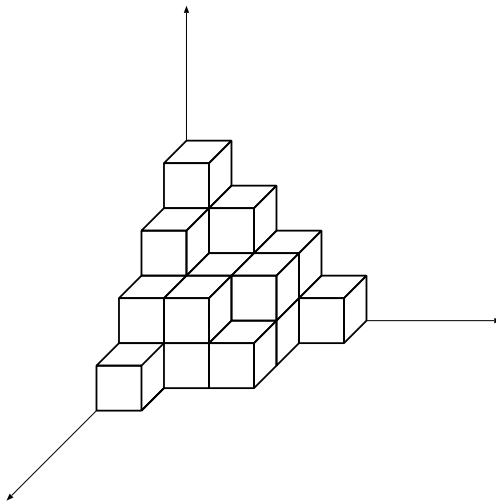


Figure 2: 3D Young diagram as complement of crystal corner

stanton sum is a 5D analogue of Nekrasov's instanton sum for 4D  $\mathcal{N} = 2$  SUSY gauge theories [7, 8]. The 4D instanton sum is a statistical sum over ordinary partitions (or “colored” partitions in the case of  $SU(N)$  theory), hence a model of random partitions. Nekrasov and Okounkov [9] used such models of random partitions to re-derive the Seiberg-Witten solutions [10] of 4D  $\mathcal{N} = 2$  SUSY gauge theories. Actually, by the aforementioned method of transfer matrices, the statistical sum over plane partitions can be reorganized to a sum over partitions. This is a kind of  $q$ -deformations of 4D instanton sums. A 5D analogue of the Seiberg-Witten solution can be derived from this  $q$ -deformed instanton sum [9, 11].

In this paper, we review our recent results [12] on an integrable structure of the melting crystal model (and the 5D  $U(1)$  instanton sum) with external potentials. The partition function  $Z_p(t)$  of this model is a function of the coupling constants  $t = (t_1, t_2, \dots)$  of the external potentials. A main conclusion of these results is that  $Z_p(t)$  is, up to a simple factor, a tau function of the 1D Toda hierarchy, in other words, a tau function  $\tau_p(t, \bar{t})$  of the 2D Toda hierarchy [13] that depends only on the difference  $t - \bar{t}$  of the two sets  $t, \bar{t}$  of time variables. To derive this conclusion, we first rewrite  $Z_p(t)$  in terms of a complex fermion system. In the case of 4D instanton sum, such a fermionic representation was proposed by Nekrasov et al. [14, 9]. In the present case, we can use the aforementioned transfer matrices [2] to construct a fermionic representation. This fermionic representation, however, does not take the form of a standard fermionic representation of the (1D or 2D) Toda hierarchy [15, 16]. To resolve this problem, we derive a set of algebraic relations (referred to as “shift symmetry”) satisfied by the transfer matrices and a set of fermion bilinear forms.

(Actually, these fermion bilinear forms turn out to give a realization of “quantum torus Lie algebra”.) These algebraic relations enable us to rewrite the fermionic representation of  $Z_p(t)$  to the standard form of Toda tau functions.

In the 4D case, a similar partition function with external potentials has been studied by Marshakov and Nekrasov [17, 18]. According to their results, the 1D Toda hierarchy is also a relevant integrable structure therein. Unfortunately, our method developed for the 5D case relies heavily on the structure of quantum torus Lie algebra, which ceases to exist in the 4D setup. We shall return to this issue, along with some other issue, in the end of this paper.

This paper is organized as follows. Section 2 is a brief review of the melting crystal model and its mathematical background. Section 3 presents the fermionic formula of the partition function. The method of transfer matrices is reviewed in detail. Section 4 deals with the quantum torus Lie algebra and its shift symmetries. In Section 5, we use this symmetry to rewrite the fermionic representation of the partition function to the standard form as a Toda tau function. Section 6 is devoted to concluding remarks.

## 2 Melting crystal model

### 2.1 Young diagrams and partitions

Let us recall [19] that an ordinary 2D Young diagram is represented by an integer partition, namely, a sequence

$$\lambda = (\lambda_1, \lambda_2, \dots), \quad \lambda_1 \geq \lambda_2 \geq \dots,$$

of nonincreasing integers  $\lambda_i \in \mathbf{Z}_{\geq 0}$  with only a finite number of  $\lambda_i$ 's being nonzero.  $\lambda_i$  is the length of  $i$ -th row of the Young diagram viewed as a collection of unit squares. We shall always identify such a partition  $\lambda$  with a Young diagram. The total area of the diagram is given by the degree

$$|\lambda| = \sum_i \lambda_i$$

of the partition.

It was shown by Euler that the generating function of the number  $p(N)$  of partitions  $\lambda$  of degree  $N$  has an infinite product formula:

$$\sum_{N=0}^{\infty} p(N) q^N = \prod_{n=1}^{\infty} (1 - q^n)^{-1}, \quad (2.1)$$

where  $q$  is assumed to be in the range  $0 < q < 1$ . One can interpret this generating function as the partition function of a model of statistical mechanics,

$$Z_{2D} = \sum_{N=0}^{\infty} p(N) q^N = \sum_{\lambda} q^{|\lambda|},$$

in which each partition  $\lambda$  is assigned an energy proportional to  $|\lambda|$ , and  $q$  is related to the temperature  $T$  as  $q = e^{-\text{const.}/T}$

## 2.2 3D Young diagrams and plane partitions

A 3D Young diagram can be represented by a “plane partition”, namely, a 2D array

$$\pi = (\pi_{ij})_{i,j=1}^{\infty} = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots \\ \pi_{21} & \pi_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

of nonnegative integers  $\pi_{ij} \in \mathbf{Z}_{\geq 0}$  such that

$$\pi_{ij} \geq \pi_{i,j+1}, \quad \pi_{ij} \geq \pi_{i+1,j}.$$

$\pi_{ij}$  is the height of the stack of cubes placed at the  $(i,j)$ -th position of the plane. We shall identify such a plane partition with the corresponding 3D Young diagram. The total volume of the 3D Young diagram is given by

$$|\pi| = \sum_{i,j=1}^{\infty} \pi_{ij}.$$

As an analogue of  $p(N)$ , one can consider the number  $\text{pp}(N)$  of plane partitions  $\pi$  with  $|\pi| = N$ . The generating function of these numbers was studied by MacMahon [1] and shown to be given, again, by an infinite product:

$$\sum_{N=0}^{\infty} \text{pp}(N) q^N = \prod_{n=1}^{\infty} (1 - q^n)^{-n}. \quad (2.2)$$

The right hand side is now called the MacMahon function. In statistical mechanics, this generating function becomes the partition function

$$Z_{3D} = \sum_{N=0}^{\infty} \text{pp}(N) q^N = \sum_{\pi} q^{|\pi|}$$

of a canonical ensemble of plane partitions, in which each plane partition  $\pi$  has an energy proportional to the volume  $|\pi|$ .

We shall deform this simplest model by external potentials. To this end, we have to introduce the notion of “diagonal slices” of a plane partition.

## 2.3 Diagonal slices of 3D Young diagrams

Given a plane partition  $\pi = (\pi_{ij})_{i,j=1}^{\infty}$ , the partition

$$\pi(m) = \begin{cases} (\pi_{i,i+m})_{i=1}^{\infty} & \text{if } m \geq 0 \\ (\pi_{j-m,j})_{j=1}^{\infty} & \text{if } m < 0 \end{cases}$$

is called the  $m$ -th diagonal slice of  $\pi$ . These partitions  $\{\pi(m)\}_{m=-\infty}^{\infty}$  represent a sequence of 2D Young diagrams that are literally obtained by slicing the 3D Young diagrams (Figure3).

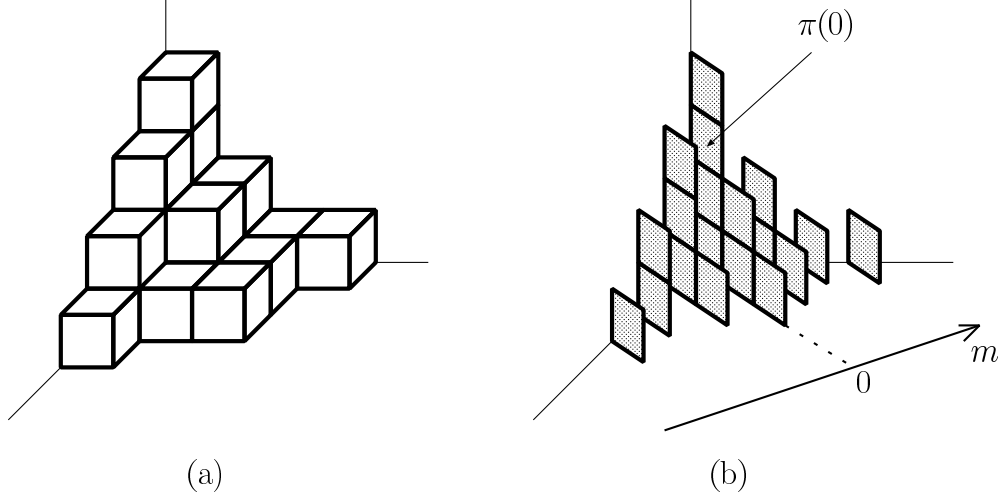


Figure 3: Diagonal slices (b) of plane partition (a)

The diagonal slices are not arbitrary but satisfy the condition [2, 3]

$$\cdots \prec \pi(-2) \prec \pi(-1) \prec \pi(0) \succ \pi(1) \succ \pi(2) \succ \cdots, \quad (2.3)$$

where “ $\succ$ ” denotes *interlacing relation*, namely,

$$\lambda = (\lambda_1, \lambda_2, \dots) \succ \mu = (\mu_1, \mu_2, \dots) \stackrel{\text{def}}{\iff} \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots.$$

Because of these interlacing relations, a pair  $(T, T')$  of semi-standard tableaux is obtained on the main diagonal slice  $\lambda = \pi(0)$  by putting “ $m+1$ ” in boxes of the skew diagram  $\pi(\pm m)/\pi(\pm(m+1))$ .

By this mapping  $\pi \mapsto (T, T')$ , the partition function  $Z_{3D}$  of the plane partitions can be converted to a triple sum over the tableau  $T, T'$  and their shape  $\lambda$ :

$$Z_{3D} = \sum_{\lambda} \sum_{T, T': \text{shape } \lambda} q^T q^{T'}, \quad (2.4)$$

where

$$q^T = \prod_{m=0}^{\infty} q^{(m+1/2)|\pi(-m)/\pi(-m-1)|},$$

$$q^{T'} = \prod_{m=0}^{\infty} q^{(m+1/2)|\pi(m)/\pi(m+1)|}.$$

By the well known combinatorial definition of the Schur functions [19], the partial sum over the semi-standard tableaux turn out to be a special value of

the Schur functions:

$$\sum_{T:\text{shape } \lambda} q^T = \sum_{T':\text{shape } \lambda} q^{T'} = s_\lambda(q^\rho), \quad (2.5)$$

where

$$q^\rho = (q^{1/2}, q^{3/2}, \dots, q^{n+1/2}, \dots).$$

Thus the partition function can be eventually rewritten as

$$Z_{3D} = \sum_{\lambda} s_\lambda(q^\rho)^2. \quad (2.6)$$

Let us note that the special value of the Schur functions has the so called Hook formula [19]

$$s_\lambda(q^\rho) = q^{n(\lambda)+|\lambda|/2} \prod_{(i,j) \in \lambda} (1 - q^{h(i,j)})^{-1}, \quad (2.7)$$

where  $(i, j)$  stands for the  $(i, j)$ -th box in the Young diagram, and  $n(\lambda)$  is given by

$$n(\lambda) = \sum_{i=1}^{\infty} (i-1)\lambda_i.$$

## 2.4 Melting crystal model with external potentials

We now deform the foregoing melting crystal model by introducing the external potentials

$$\Phi_k(\lambda, p) = \sum_{i=1}^{\infty} q^{k(p+\lambda_i-i+1)} - \sum_{i=1}^{\infty} q^{k(-i+1)}$$

with coupling constants  $t_k$ ,  $k = 1, 2, 3, \dots$ , on the main diagonal slice  $\lambda = \pi(0)$ . The right hand side of the definition of  $\Phi_k(\lambda, p)$  is understood to be a finite sum (hence a rational function of  $q$ ) by cancellation of terms between the two sums:

$$\Phi_k(\lambda, p) = \sum_{i=1}^{\infty} (q^{k(p+\lambda_i-i+1)} - q^{k(p-i+1)}) + q^k \frac{1 - q^{pk}}{1 - q^k}.$$

The partition function of the deformed model reads

$$Z_p(t) = \sum_{\pi} q^{|\pi|} e^{\Phi(t, \pi(0), p)}, \quad (2.8)$$

where

$$\Phi(t, \lambda, p) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda, p).$$

We can repeat the previous calculations in this setting to rewrite the new partition function  $Z_p(t)$  as

$$Z_p(t) = \sum_{\lambda} s_{\lambda}(q^{\rho})^2 q^{\Phi(t, \lambda, p)}. \quad (2.9)$$

Modifying this partition function slightly, we obtain the instanton sum of 5D  $\mathcal{N} = 1$  SUSY  $U(1)$  Yang-Mills theory [6]:

$$Z_p(t) = \sum_{\pi} q^{|\pi|} Q^{\pi(0)} e^{\Phi(t, \pi(0), p)} = \sum_{\lambda} s_{\lambda}(q^{\rho})^2 Q^{|\lambda|} e^{\Phi(t, \lambda, p)}. \quad (2.10)$$

$q$  and  $Q$  are related to physical parameters  $R, \Lambda, \hbar$  of 5D Yang-Mills theory as

$$q = e^{-R\hbar}, \quad Q = (R\Lambda)^2.$$

The external potentials represent the contribution of Wilson loops along the fifth dimension [20]. In this sense,  $Z_p(t)/Z_p(0)$  is a generating function of correlation functions of those Wilson loop operators.

Our goal is to show that the partition function  $Z_p(t)$  is, up to a simple factor, the tau function of the (1D) Toda hierarchy. To this end, we now consider a fermionic representation of this partition function.

### 3 Fermionic formula of partition function

#### 3.1 Complex fermion system

Let  $\psi(z)$  and  $\psi^*(z)$  denote complex 2D fermion fields

$$\psi(z) = \sum_{m=-\infty}^{\infty} \psi_m z^{-m-1}, \quad \psi^*(z) = \sum_{m=-\infty}^{\infty} \psi_m^* z^{-m}.$$

The Fourier modes  $\psi_m$  and  $\psi_m^*$  of  $\psi(z)$  and  $\psi^*(z)$  satisfy the anti-commutation relations

$$\{\psi_m, \psi_n^*\} = \delta_{m+n, 0}, \quad \{\psi_m, \psi_n\} = \{\psi_m^*, \psi_n^*\} = 0.$$

The Fock space  $F$  splits into charge  $p$  subspaces  $F_p$ :

$$F = \bigoplus_{p=-\infty}^{\infty} F_p.$$

The charge  $p$  subspace  $F_p$  has a unique normalized ground state (charge  $p$  vacuum)  $|p\rangle$  and an orthonormal basis  $|\lambda; p\rangle$  labeled by partitions  $\lambda$ .  $|p\rangle$  is characterized by the vacuum condition

$$\psi_m |p\rangle = 0 \quad \text{for } m \geq -p, \quad \psi_m^* |p\rangle = 0 \quad \text{for } m \geq p+1.$$



If the partition is of the form  $\lambda = (\lambda_1, \dots, \lambda_n, 0, 0, \dots)$ , the associated element  $|\lambda; p\rangle$  of the basis is obtained from  $|p\rangle$  by the action of fermion operators as

$$|\lambda; p\rangle = \psi_{-(p+\lambda_1-1)-1} \cdots \psi_{-(p+\lambda_n-n)-1} \psi_{(p-n)+1}^* \cdots \psi_{(p-1)+1}^* |p\rangle.$$

They are orthonormal in the sense that their inner products have the normalized values

$$\langle \lambda; p | \mu; q \rangle = \delta_{pq} \delta_{\lambda\mu}.$$

### 3.2 $U(1)$ current and fermionic representation of tau function

The  $U(1)$  current  $J(z)$  of the complex fermion system is defined as

$$J(z) = :\psi(z)\psi^*(z): = \sum_{k=-\infty}^{\infty} J_m z^{-m-1},$$

where  $: \ :$  denotes the normal ordering with respect to the vacuum  $|0\rangle$ :

$$:\psi_m \psi_n^*: = \psi_m \psi_n^* - \langle 0 | \psi_m \psi_n^* | 0 \rangle.$$

The Fourier modes

$$J_m = \sum_{n=-\infty}^{\infty} :\psi_{m-n} \psi_n^*:$$

of  $J(z)$  satisfy the commutation relations

$$[J_m, J_n] = m \delta_{m+n, 0} \quad (3.1)$$

of the  $A_\infty$  Heisenberg algebra, and play the role of “Hamiltonians” in the usual fermionic formula of the KP and 2D Toda hierarchies [15, 16]. For the case of tau functions  $\tau(t, \bar{t})$ ,  $t = (t_1, t_2, \dots)$ ,  $\bar{t} = (\bar{t}_1, \bar{t}_2, \dots)$ , of the 2D Toda hierarchy, the fermionic formula reads

$$\tau_p(t, \bar{t}) = \langle p | \exp\left(\sum_{m=1}^{\infty} t_m J_m\right) g \exp\left(-\sum_{m=1}^{\infty} \bar{t}_m J_{-m}\right) | p \rangle \quad (3.2)$$

where  $g$  is an element of the infinite dimensional Clifford group  $GL(\infty)$ .

### 3.3 Fermionic representation of $Z_p(t)$

The partition function  $Z_p(t)$  of the deformed melting crystal has a fermionic representation of the form

$$Z_p(t) = \langle p | G_+ e^{H(t)} G_- | p \rangle. \quad (3.3)$$

Let us explain the constituents of this formula along with an outline of the derivation of this formula.

$H(t)$  is the linear combination

$$H(t) = \sum_{k=1}^{\infty} t_k H_k$$

of the “Hamiltonians”

$$H_k = \sum_{n=-\infty}^{\infty} q^{kn} : \psi_{-n} \psi_n^* :.$$

The aforementioned basis elements  $|\lambda; p\rangle$  of the Fermion Fock space turn out to be eigenvectors of these Hamiltonians. The eigenvalues are nothing but the the potential functions  $\Phi_k(\lambda, p)$ :

$$H_k |\lambda; p\rangle = \Phi_k(\lambda, p) |\lambda; p\rangle. \quad (3.4)$$

$G_{\pm}$  are  $GL(\infty)$  elements of the special form

$$G_{\pm} = \exp \left( \sum_{k=1}^{\infty} \frac{q^{k/2}}{k(1-q^k)} J_{\pm k} \right).$$

Since the numerical factors  $q^{k/2}/(1-q^k)$  in this definition can be expanded as

$$\frac{q^{k/2}}{1-q^k} = \sum_{m=-\infty}^{-1} q^{-k(m+1/2)} = \sum_{m=0}^{\infty} q^{k(m+1/2)},$$

one can factorize these operators as

$$G_+ = \prod_{m=-\infty}^{-1} \Gamma_+(m), \quad G_- = \prod_{m=0}^{\infty} \Gamma_-(m), \quad (3.5)$$

where

$$\Gamma_{\pm}(m) = \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} q^{\mp k(m+1/2)} J_{\pm k} \right).$$

These  $\Gamma_{\pm}(m)$ ’s are a specialization of the so called vertex operators

$$V_{\pm}(z) = \exp \left( \sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k} \right)$$

for bosonization of the complex fermions.

Following the idea of Okounkov and Reshetikhin [2], we now consider  $m$  as a fictitious “time” variable. A plane partition then may be thought of as the

“path” (or “world volume”) of discrete time evolutions of a partition  $\lambda$  that starts from the empty partition  $\emptyset = (0, 0, \dots)$  at infinite past and ends again in  $\emptyset$  at infinite future (see Figure 3). The vertex operators  $\Gamma_{\pm}(m)$  play the role of transfer matrices between neighboring diagonal slices.

The vertex operators  $\Gamma_{\pm}(m)$  act on the aforementioned orthonormal bases  $|\lambda; p\rangle$  and  $\langle\lambda; p|$  as

$$\langle\lambda; p|\Gamma_+(m) = \sum_{\mu \succ \lambda} \langle\mu; p|q^{-(m+1/2)(|\mu|-|\lambda|)} \quad (3.6)$$

for  $m = -1, -2, \dots$  and

$$\Gamma_-(m)|\lambda; p\rangle = \sum_{\mu \succ \lambda} q^{(m+1/2)(|\mu|-|\lambda|)} |\mu; p\rangle \quad (3.7)$$

for  $m = 0, 1, \dots$  [2, 3]. The right hand side of these formulas give a linear combination of all possible time evolutions of the  $m$ -th slice  $\lambda = \pi(m)$  at the next time. The weight  $q^{\mp(m+1/2)(|\mu|-|\lambda|)}$  of each state on the right hand side are exactly the factors assigned to the boxes of  $\pi(m)/\pi(m \mp 1)$  in the definition of the weights  $q^T, q^{T'}$  that appear in the combinatorial formula (2.4).

Since  $G_{\pm}$  are products of these slice-to-slice “transfer matrices”,  $\langle p|G_+$  and  $G_-|p\rangle$  become linear combinations of the states  $\langle\lambda; p|$  and  $|\lambda; p\rangle$  that evolve from the ground states  $\langle p|$  and  $|p\rangle$  at  $m = \mp\infty$ . By what we have seen above, the weights of  $\langle\lambda; p|$  and  $|\lambda; p\rangle$  in these linear combinations are given by the partial sums of  $q^T$  and  $q^{T'}$  over all semi-standard tableaux  $T$  and  $T'$  of shape  $\lambda$ , namely, the special value  $s_{\lambda}(q^{\rho})$  of the Schur function. Thus  $\langle p|G_+$  and  $G_-|p\rangle$  can be expressed as

$$\langle p|G_+ = \sum_{\lambda} \sum_{T: \text{shape } \lambda} q^T \langle\lambda; p| = \sum_{\lambda} s_{\lambda}(q^{\rho}) \langle\lambda; p|, \quad (3.8)$$

$$G_-|p\rangle = \sum_{\lambda} \sum_{T': \text{shape } \lambda} q^{T'} |\lambda; p\rangle = \sum_{\lambda} s_{\lambda}(q^{\rho}) |\lambda; p\rangle. \quad (3.9)$$

The expectation value of  $e^{H(t)}$  with respect to these states yields the fermionic representation (3.3) of the partition function  $Z_p(t)$ .

The fermionic representation (3.3) is apparently different from the fermionic formula (3.2) of tau functions of the 2D Toda hierarchy. To show that  $Z_p(t)$  is indeed a tau function, we have to rewrite (3.3) to the form of (3.2). This is the place where the quantum torus Lie algebra joins the game.

## 4 Quantum torus Lie algebra

### 4.1 Fermionic realization of quantum torus Lie algebra

Let  $V_m^{(k)}$  ( $k = 0, 1, \dots$ ,  $m \in \mathbf{Z}$ ) denote the following fermion bilinear forms:

$$\begin{aligned} V_m^{(k)} &= q^{-km/2} \sum_{n=-\infty}^{\infty} q^{kn} : \psi_{m-n} \psi_n^* : \\ &= q^{k/2} \oint \frac{dz}{2\pi i} z^m : \psi(q^{k/2} z) \psi^*(q^{-k/2} z) : \end{aligned}$$

Note that

$$J_m = V_m^{(0)}, \quad H_k = V_0^{(k)}.$$

Actually,  $V_m^{(k)}$  coincides with Okounkov and Pandharipande's operator  $\mathcal{E}_m(z)$  [21, 22] specialized to  $z = q^k$ . As they found for  $\mathcal{E}_m(z)$ , our  $V_m^{(k)}$ 's satisfy the commutation relations

$$[V_m^{(k)}, V_n^{(l)}] = (q^{(lm-kn)/2} - q^{(kn-lm)/2})(V_{m+n}^{(k+l)} - \delta_{m+n,0} \frac{q^{k+l}}{1-q^{k+l}}). \quad (4.1)$$

This is a (central extension of)  $q$ -deformation of the Poisson algebra of functions on a 2-torus. We refer to this Lie algebra as “quantum torus Lie algebra”. More precisely, a full quantum torus Lie algebra should contain elements for  $k < 0$  as well; for several reasons, we shall not include those elements.

### 4.2 Shift symmetry among basis of quantum torus Lie algebras

The following relations, which we call “shift symmetry”, play a central role in identifying  $Z_p(t)$  as a tau function:

$$G_- G_+ \left( V_m^{(k)} - \delta_{m,0} \frac{q^k}{1-q^k} \right) (G_- G_+)^{-1} = (-1)^k \left( V_{m+k}^{(k)} - \delta_{m+k,0} \frac{q^k}{1-q^k} \right) \quad (4.2)$$

These relations are derived as follows.

Let us recall that the fermion fields  $\psi(z), \psi^*(z)$  transform under adjoint action by  $J_{\pm k}$ 's as

$$\exp\left(\sum_{k=1}^{\infty} c_k J_{\pm k}\right) \psi(z) \exp\left(-\sum_{k=1}^{\infty} c_k J_{\pm k}\right) = \exp\left(\sum_{k=1}^{\infty} c_k z^{\pm k}\right) \psi(z), \quad (4.3)$$

$$\exp\left(\sum_{k=1}^{\infty} c_k J_{\pm k}\right) \psi^*(z) \exp\left(-\sum_{k=1}^{\infty} c_k J_{\pm k}\right) = \exp\left(-\sum_{k=1}^{\infty} c_k z^{\pm k}\right) \psi^*(z). \quad (4.4)$$

By letting  $c_k = q^{k/2}/(1 - q^k)$ , the exponential operators in these formulas turn into  $G_{\pm}$ , so that we have the operator identities

$$G_+ \psi(z) G_+^{-1} = (q^{1/2} z; q)_{\infty}^{-1} \psi(z), \quad (4.5)$$

$$G_+ \psi^*(z) G_+^{-1} = (q^{1/2} z; q)_{\infty} \psi^*(z), \quad (4.6)$$

$$G_- \psi(z) G_-^{-1} = (q^{1/2} z^{-1}; q)_{\infty}^{-1} \psi(z), \quad (4.7)$$

$$G_- \psi^*(z) G_-^{-1} = (q^{1/2} z^{-1}; q)_{\infty} \psi^*(z), \quad (4.8)$$

where  $(z; q)_{\infty}$  denotes the standard  $q$ -factorial symbol

$$(z; q)_{\infty} = \prod_{n=0}^{\infty} (1 - zq^n).$$

We use these operator identities to derive transformation of the fermion bilinear forms  $:\psi(q^{k/2} z) \psi^*(q^{-k/2} z):$  under conjugation by  $G_{\pm}$ . Since

$$:\psi(q^{k/2} z) \psi^*(q^{-k/2} z): = -\psi^*(q^{-k/2} z) \psi(q^{k/2} z) + \frac{q^{k/2}}{(1 - q^k)z},$$

let us first consider  $\psi^*(q^{-k/2} z) \psi(q^{k/2} z)$ . Under conjugation by  $G_+$ , it transforms as

$$\begin{aligned} G_+ \psi^*(q^{-k/2} z) \psi(q^{k/2} z) G_+^{-1} \\ = \frac{(q^{1/2} \cdot q^{-k/2} z; q)_{\infty}}{(q^{1/2} \cdot q^{k/2} z; q)_{\infty}} \psi^*(q^{-k/2} z) \psi(q^{k/2} z) \\ = \prod_{m=1}^k (1 - zq^{(k+1)/2-m}) \psi^*(q^{-k/2} z) \psi(q^{k/2} z). \end{aligned}$$

This implies that

$$\begin{aligned} G_+ \left( :\psi(q^{k/2} z) \psi^*(q^{-k/2} z): - \frac{q^{k/2}}{(1 - q^k)z} \right) G_+^{-1} \\ = \prod_{m=1}^k (1 - q^{(k+1)/2-m} z) \left( :\psi(q^{k/2} z) \psi^*(q^{-k/2} z): - \frac{q^{k/2}}{(1 - q^k)z} \right). \quad (4.9) \end{aligned}$$

In much the same way, we can derive a similar transformation under conjugation by  $G_-$ . In this case, it is more convenient to rewrite the result as follows:

$$\begin{aligned} G_-^{-1} \left( :\psi(q^{k/2} z) \psi^*(q^{-k/2} z): - \frac{q^{k/2}}{(1 - q^k)z} \right) G_- \\ = \prod_{m=1}^k (1 - q^{-(k+1)/2+m} z^{-1}) \left( :\psi(q^{k/2} z) \psi^*(q^{-k/2} z): - \frac{q^{k/2}}{(1 - q^k)z} \right) \quad (4.10) \end{aligned}$$

We note here that the prefactors on the right hand side of the last two equations are related as

$$\prod_{m=1}^k (1 - q^{(k+1)/2-m} z) = (-z)^k \prod_{m=1}^k (1 - q^{-(k+1)/2+m} z^{-1}).$$

Accounting for this simple, but significant relation, we can derive the identity

$$\begin{aligned} G_- G_+ \left( : \psi(q^{k/2} z) \psi^*(q^{-k/2} z) : - \frac{q^{k/2}}{(1-q^k)z} \right) (G_+ G_-)^{-1} \\ = (-z)^k \left( : \psi(q^{k/2} z) \psi^*(q^{-k/2} z) : - \frac{q^{k/2}}{(1-q^k)z} \right). \end{aligned} \quad (4.11)$$

The shift symmetry (4.2) follows immediately from this identity.

When  $m = 0$  and  $m = -k$ , (4.2) takes the particular form

$$G_- G_+ \left( V_0^{(k)} - \frac{q^k}{1-q^k} \right) (G_- G_+)^{-1} = (-1)^k V_k^{(k)}, \quad (4.12)$$

$$(G_- G_+)^{-1} \left( V_0^{(k)} - \frac{q^k}{1-q^k} \right) G_- G_+ = (-1)^k V_{-k}^{(k)}. \quad (4.13)$$

It is these identities that we shall use to convert the fermionic representation of  $Z_p(t)$  to the standard fermionic formula of tau functions.

## 5 Integrable structure of melting crystal model

### 5.1 Partition function as tau function of 2D Toda hierarchy

Let us split the operator  $G_+ e^{H(t)} G_-$  in (3.3) into three pieces as

$$\begin{aligned} G_+ e^{H(t)} G_- &= G_+ e^{H(t)/2} e^{H(t)/2} G_- \\ &= G_+ e^{H(t)/2} G_+^{-1} \cdot G_+ G_- \cdot G_-^{-1} e^{H(t)/2} G_- \end{aligned}$$

and use the special cases (4.12) and (4.13) of the shift symmetry to rewrite those pieces.

To this end, it is convenient to rewrite (4.12) and (4.13) as

$$\begin{aligned} G_+ \left( H_k - \frac{q^k}{1-q^k} \right) G_+^{-1} &= (-1)^k G_-^{-1} V_k^{(k)} G_-, \\ G_-^{-1} \left( H_k - \frac{q^k}{1-q^k} \right) G_- &= (-1)^k G_+ V_{-k}^{(k)} G_+^{-1}. \end{aligned}$$

Though the operators  $V_{\pm k}^{(k)}$  on the right hand side are unfamiliar in the theory of integrable hierarchies, we can convert them to the familiar ‘‘Hamiltonians’’  $J_{\pm k} = V_{\pm k}^{(0)}$  of the Toda hierarchies as

$$q^{W/2} V_k^{(k)} q^{-W/2} = V_k^{(0)} = J_k, \quad q^{-W/2} V_{-k}^{(k)} q^{W/2} = V_{-k}^{(0)} = J_{-k}, \quad (5.1)$$

where  $W$  is a special element of  $W_\infty$  algebra:

$$W = W_0^{(3)} = \sum_{n=-\infty}^{\infty} n^2 : \psi_{-n} \psi_n^* :$$

We thus eventually obtain the relations

$$G_+ \left( H_k - \frac{q^k}{1-q^k} \right) G_+^{-1} = (-1)^k G_-^{-1} q^{-W/2} J_k q^{W/2} G_-, \quad (5.2)$$

$$G_-^{-1} \left( H_k - \frac{q^k}{1-q^k} \right) G_- = (-1)^k G_+ q^{W/2} J_{-k} q^{-W/2} G_+^{-1} \quad (5.3)$$

between  $H_k$ 's and  $J_{\pm k}$ 's.

By these relations,  $G_+ e^{H(t)/2} G_+^{-1}$  can be calculated as

$$\begin{aligned} & G_+ e^{H(t)/2} G_+^{-1} \\ &= \exp \left( \sum_{k=1}^{\infty} \frac{t_k q^k}{2(1-q^k)} \right) G_-^{-1} q^{-W/2} \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^k t_k}{2} J_k \right) q^{W/2} G_-. \end{aligned}$$

A similar expression can be derived for  $G_-^{-1} e^{H(t)} G_-$  as well. We can thus rewrite  $G_+ e^{H(t)} G_-$  as

$$\begin{aligned} G_+ e^{H(t)} G_- &= \exp \left( \sum_{k=1}^{\infty} \frac{t_k q^k}{1-q^k} \right) G_-^{-1} q^{-W/2} \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^k t_k}{2} J_k \right) \times \\ &\quad \times g \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^k t_k}{2} J_{-k} \right) q^{-W/2} G_+^{-1} \end{aligned}$$

where

$$g = q^{W/2} (G_- G_+)^2 q^{W/2}. \quad (5.4)$$

The partition function  $Z_p(t)$  is given by the expectation value of this operator with respect to  $\langle p|$  and  $|p\rangle$ . Since the action by the leftmost and rightmost pieces of  $g$  yields only a scalar multiplier to  $\langle p|$ ,  $|p\rangle$  as

$$\langle p| G_-^{-1} q^{-W/2} = q^{-p(p+1)(2p+1)/12} \langle p|, \quad (5.5)$$

$$q^{-W/2} G_+^{-1} |p\rangle = q^{-p(p+1)(2p+1)/12} |p\rangle, \quad (5.6)$$

$Z_p(t)$  can be expressed as

$$\begin{aligned} Z_p(t) &= \exp \left( \sum_{k=1}^{\infty} \frac{t_k q^k}{1-q^k} \right) q^{-p(p+1)(2p+1)/6} \times \\ &\quad \times \langle p| \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^k t_k}{2} J_k \right) g \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^k t_k}{2} J_{-k} \right) |p\rangle. \end{aligned} \quad (5.7)$$

The expectation value  $\langle p| \cdots |p\rangle$  takes exactly the form of (3.2). Thus, up to the simple prefactor,  $Z_p(t)$  is essentially a tau function of the 2D Toda hierarchy. Thus we find that an integrable structure behind the melting crystal model is the 2D Toda hierarchy. This is, however, not the end of the story.

## 5.2 1D Toda hierarchy as true integrable structure

The foregoing calculation is based on the splitting

$$G_+ e^{H(t)} G_- = G_+ e^{H(t)/2} G_+^{-1} \cdot G_+ G_- \cdot G_-^{-1} e^{H(t)/2} G_-.$$

Actually, we could have started from a different splitting of  $G_+ e^{H(t)} G_-$ , e.g.,

$$G_+ e^{H(t)} G_- = G_+ e^{H(t)} G_+^{-1} \cdot G_+ G_- = G_+ G_- \cdot G_-^{-1} e^{H(t)} G_-.$$

This leads to another set of expressions of  $Z_p(t)$  in which only the  $\langle p | \cdots | p \rangle$  part is different. We thus have the following three different expressions for this part:

$$\begin{aligned} & \langle p | \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^k t_k}{2} J_k\right) g \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^k t_k}{2} J_{-k}\right) | p \rangle \\ &= \langle p | \exp\left(\sum_{k=1}^{\infty} (-1)^k t_k J_k\right) g | p \rangle \\ &= \langle p | g \exp\left(\sum_{k=1}^{\infty} (-1)^k t_k J_{-k}\right) | p \rangle. \end{aligned} \quad (5.8)$$

These identities of the expectation values can be directly derived from the operator identities

$$J_k g = g J_{-k}, \quad k = 1, 2, 3, \dots \quad (5.9)$$

satisfied by  $g$ . (These operator identities themselves are a consequences of the shift symmetry of  $V_m^{(k)}$ 's.) Generally speaking, this kind of operator identities imply symmetry constraints on the tau functions [23, 24]; in the present case, the constraints read

$$\frac{\partial}{\partial t_k} \tau_p(t, \bar{t}) + \frac{\partial}{\partial \bar{t}_k} \tau_p(t, \bar{t}) = 0, \quad k = 1, 2, 3, \dots \quad (5.10)$$

In other words, the tau function is a function of  $t - \bar{t}$ ,

$$\tau_p(t, \bar{t}) = \tau_p(t - \bar{t}, 0) = \tau_p(0, \bar{t} - t),$$

and reduces to a tau function  $\tau_p(t)$  of the 1D Toda hierarchy that has a single series of time variables  $t = (t_1, t_2, \dots)$  rather than the two series of the 2D Toda hierarchy. Thus the 1D Toda hierarchy eventually turns out to be an underlying integrable structure of the deformed melting crystal model.

The same conclusion can be derived for the instanton sum (2.10) of 5D SUSY  $U(1)$  Yang-Mills theory. It has a fermionic representation of the form

$$Z_p(t) = \langle p | G_+ Q^{L_0} e^{H(t)} G_- | p \rangle \quad (5.11)$$

where  $L_0$  is a special element of the Virasoro algebra:

$$L_0 = \sum_{n=-\infty}^{\infty} n : \psi_{-n} \psi_n^* :$$



One can repeat almost the same calculations as the previous case to convert  $Z_p(t)$  to the form of (5.7). The counterpart of  $g$  is given by

$$g = q^{W/2} G_- G_+ Q^{L_0} G_- G_+ q^{W/2}, \quad (5.12)$$

which, too, satisfy the reduction conditions (5.9) to the 1D Toda hierarchy. Thus a relevant integrable structure is again the 1D Toda hierarchy.

## 6 Concluding remarks

### 6.1 Problems on 4D instanton sum

In deriving the instanton sum (2.10), 5D space-time is partially compactified in the fifth dimension as  $\mathbf{R}^4 \times S^1$ . The parameter  $R$  is the radius of  $S^1$ . Therefore, letting  $R \rightarrow 0$  amounts to 4D limit.

Unfortunately, it is not straightforward to achieve such a 4D limit in the present setup. Firstly, the 5D instanton sum with external potentials does not have a reasonable limit as  $R \rightarrow 0$ . Any naive prescription letting  $R \rightarrow 0$  yields a result in which  $t$  dependence disappears or becomes trivial [12]. Secondly, the shift symmetry of the quantum torus Lie algebra ceases to exist in the limit as  $q = e^{-R\hbar} \rightarrow 1$ . Speaking more precisely, the quantum torus Lie algebra itself turns into a  $W_\infty$  algebra in this limit, but no analogue of shift symmetry (4.2) is known for the latter case. For these reasons, the 4D case has to be studied independently.

The 4D instanton sum [7, 8], too, is a sum over partitions. Moreover, this statistical sum has a fermionic representation [14, 9]. Marshakov and Nekrasov [17, 18] further introduced external potentials therein. Actually, the 4D instanton sum for  $U(1)$  gauge theory is almost identical to the generating function of Gromov-Witten invariants of  $\mathbf{CP}^1$  [21, 22]. This can be most clearly seen in the fermionic representation of these generating functions, which reads

$$Z_p^{4D}(t) = \langle p | e^{J_1/\hbar} \exp\left(\sum_{k=1}^{\infty} t_k \frac{\mathcal{P}_{k+1}}{k+1}\right) e^{J_{-1}/\hbar} | p \rangle, \quad (6.1)$$

where  $\mathcal{P}_k$ 's are fermion bilinear forms introduced by Okounkov and Pandharipande for a fermionic representation of (absolute) Gromov-Witten invariants of  $\mathbf{CP}^1$  [21]. As regards these Gromov-Witten invariants (in other words, correlation functions of the topological  $\sigma$  model), it has been known for years [25, 26, 27, 28, 29] that a relevant integrable structure is the 1D Toda hierarchy.

Thus the 1D Toda hierarchy is expected to be the integrable structure of the 4D instanton sum as well. This has been confirmed by Marshakov and Nekrasov in detail [17, 18]. What is still missing, however, is a formula like (5.7) that directly connects  $Z_p^{4D}(t)$  with the standard fermionic formula (3.2) of the tau function. Finding a 4D analogue of (5.7) is thus an intriguing open problem. This issue is also closely related to the fate of shift symmetry (4.2) in the  $q \rightarrow 1$  limit.

## 6.2 Relation to topological strings

Our results are directly or indirectly connected with some aspects of topological strings as well.

1. According to the theory of topological vertex [5], the partition function  $Z_p(t)$  of the deformed melting crystal model has another interpretation as the  $A$ -model topological string amplitude for the toric Calabi-Yau threefold  $\mathcal{O} \oplus \mathcal{O}(-2) \rightarrow \mathbf{CP}^1$ . In this interpretation,  $q$  and  $Q$  are parametrized by the string coupling constant  $g_{\text{st}}$  and the Kähler volume  $a$  of  $\mathbf{CP}^1$  as

$$q = e^{-g_{\text{st}}}, \quad Q = e^{-a}.$$

Specializing the value of  $t$  leads to several interesting observations [12].

2. A generating function of the two-legged topological vertex  $W_{\lambda\mu} \sim c_{\lambda\mu\bullet}$  is known to give a tau function of the 2D Toda hierarchy [30]. In the fermionic representation (3.2), this amounts to the case where

$$g = q^{W/2} G_+ G_- q^{W/2}. \quad (6.2)$$

(Actually, for complete agreement with the usual convention, we have to replace  $W$  with

$$K = \sum_{n=-\infty}^{\infty} \left(n - \frac{1}{2}\right)^2 : \psi_{-n} \psi_n^* :,$$

but this is not a serious problem. The difference can be absorbed by rescaling  $t_k$ 's.) Let us stress that this  $GL(\infty)$  element does not satisfy the reduction condition (5.9) to the 1D Toda hierarchy.

3. A generating function of double Hurwitz numbers for coverings of  $\mathbf{CP}^1$  gives yet another type of tau function of the 2D Toda hierarchy [31]. Actually, the  $GL(\infty)$  element for the fermionic representation is given by

$$g = q^{W/2}. \quad (6.3)$$

(More precisely, as in the previous case,  $W$  has to be replaced by  $K$ , but the difference is again irrelevant.) In this case, the reduction condition (5.9) to the 1D Toda hierarchy is not satisfied, but the operator identities (5.1) imply that another set of reduction conditions are hidden behind (see below).

## 6.3 Constraints and quantum torus Lie algebra

As a consequence of the shift symmetry of  $V_m^{(k)}$ 's, the  $GL(\infty)$  elements  $g$  of the aforementioned models of topological strings turn out to satisfy some algebraic relations other than (5.9). According to general results on constraints of the 2D Toda hierarchy [23, 24], such relations imply the existence of constraints on the tau functions and the Lax and Orlov-Schulman operators. Those constraints inherit the structure of the quantum torus Lie algebra. Let us illustrate this observation for the case of double Hurwitz numbers over  $\mathbf{CP}^1$ .

The  $GL(\infty)$  element  $g = q^{W/2}$  for this case satisfies the operator identities

$$J_k g = g V_k^{(k)}, \quad g J_{-k} = V_{-k}^{(k)} g \quad (6.4)$$

as a consequence of (5.1). These identities can be converted to the constraints

$$L = q^{1/2} q^{\bar{M}} \bar{L}, \quad \bar{L}^{-1} = q^{-1/2} q^M L^{-1} \quad (6.5)$$

on the Lax and Orlov-Schulman operators  $L, M, \bar{L}, \bar{M}$  of the 2D Toda hierarchy. Emergence of the exponential operators  $q^M$  and  $q^{\bar{M}}$  is a manifestation of the quantum torus Lie algebra. To see this, let us recall that the Lax and Orlov-Schulman operators satisfy the (twisted) canonical commutation relations

$$[L, M] = L, \quad [\bar{L}, \bar{M}] = \bar{L}. \quad (6.6)$$

This implies that the monomials  $q^{-km/2} L^m q^{kM}$  and  $q^{-km/2} \bar{L}^m q^{k\bar{M}}$  of  $L, q^M, \bar{L}, q^{\bar{M}}$  give two copies of realizations of the quantum torus Lie algebra.

The constraints (6.5) are remarkably similar to the “string equations”

$$L = \bar{M} \bar{L}, \quad \bar{L}^{-1} = M L^{-1} \quad (6.7)$$

of  $c = 1$  strings at self-dual radius [32, 33, 23, 24]. A relevant algebraic structure of these string equations is the  $W_\infty$  algebra. Thus (6.5) may be thought of as  $q$ -deformations of these  $W$ -algebraic constraints.

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